# Transition to Mathematical Proofs
## Chapter 5 - Complex Numbers Assignment Solutions

**Question 1.** Similar to how we obtained the double-angle formulae in the notes, use the Euler equation to show the two angle-sum formulae hold:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha;$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

**Solution 1.** We will use the following law of exponents:

$$e^{i(\alpha+\beta)} = e^{i\alpha} \cdot e^{i\beta}.$$

Using the Euler equation on $e^{i(\alpha+\beta)}$, we have

$$e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

Using the Euler equation on the other side and FOILing, gives us

$$e^{i\alpha} \cdot e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) =$$

$$\cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta + i^2\sin\alpha\sin\beta =$$

$$(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta).$$

Since $e^{i(\alpha+\beta)} = e^{i\alpha} \cdot e^{i\beta}$, we can equate the above results to get

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) =$$

$$(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta).$$

Equating their real parts, we get

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Equating their imaginary parts, we get

$$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta.$$

□

---

**Question 2.**

(a) Show that $|z| = \text{Re}(z)$ if and only if $z$ is a non-negative real number.

(b) Show that $(\overline{z})^2 = z^2$ if and only if $z$ is purely real or purely imaginary (i.e., its real part is 0).

**Solution 2a.** To prove the above biconditional statement, we will prove the following two conditional statements: "If $|z| = \text{Re}(z)$, then $z$ is a non-negative real number" and "If $z$ is a non-negative real number, then $|z| = \text{Re}(z)$"

For the first conditional statement, we assume that $|z| = \text{Re}(z)$. Writing $z = a + bi$, we have that

$$\sqrt{a^2+b^2} = a.$$

Squaring both side, we get that $a^2 + b^2 = a^2$ and thus $b^2 = 0$. Thus, we can conclude that $b = 0$ and so $z = a + 0i$ is a real number. To prove that it is non-negative, we note that since $z = a = \text{Re}(z) = |z|$ and $|z| \geq 0$, then $z \geq 0$. Thus, $z$ is a real, non-negative number.

For the second conditional statement, we assume that $z$ is a non-negative real number. Thus, $z = a + 0i$ with $a \geq 0$. Thus,

$$|z| = |a + 0i| = \sqrt{a^2 + 0^2} = \sqrt{a^2} = |a|.$$

Since $a \geq 0$, then $|a| = a$. Thus, $|z| = |a| = a = \text{Re}(z)$, as desired.

Having proven both conditional statements, the original biconditional statement "$|z| = \text{Re}(z)$ if and only if $z$ is a non-negative real number" is also true.

□

**Solution 2b.** To prove the above biconditional statement, we will prove the following two conditional statements: "If $(\overline{z})^2 = z^2$, then $z$ is purely real or purely imaginary" and "If $z$ is purely real or purely imaginary, then $(\overline{z})^2 = z^2$"

For the first conditional statement, we assume that $z = a + bi$ satisfies $(\overline{z})^2 = z^2$. Thus,

$$(a - bi)^2 = (a + bi)^2.$$

Expanding both sides, we get

$$a^2 - 2abi - b^2 = a^2 + 2abi - b^2.$$

Simplifying, we obtain that $-2abi = 2abi$, which is equivalent to $4abi = 0$. Dividing by $4i$, we get $ab = 0$, and thus $a = 0$ or $b = 0$, giving two cases. If $a = 0$, then $z = 0 + bi$ is purely imaginary. If $b = 0$, then $z = a + 0i$ is purely real.

For the second conditional statement, we will assume that $z = a+bi$ is purely real or purely imaginary, giving us two cases. For the first case, $z$ is purely real. Since $z$ is real, then $\overline{z} = z$. Thus, $(\overline{z})^2 = z^2$, as desired. If $z$ is purely imaginary, then $z = 0 + bi$ and thus $\overline{z} = \overline{bi} = -bi$. Thus,

$$(\overline{z})^2 = (-bi)^2 = (-1)^2(bi)^2 = z^2,$$

as desired. In either case, $(\overline{z})^2 = z^2$.

Having proven both conditional statements, the biconditional statement "$(\overline{z})^2 = z^2$ if and only if $z$ is purely real or purely imaginary" is also true.

□

---

**Question 3.** The modulus of a complex number is, in many ways, a generalization of the absolute value of a real number. Here, we give another property of the modulus that the absolute value of a real number already enjoys.
If $z, w \in \mathbb{C}$, show that

$$|z \cdot w| = |z| \cdot |w|$$

in the following two ways:

(a) By using the cartesian form $z = a + bi$ and $w = c + di$ for the complex numbers $z$ and $w$.

(b) By using the polar form $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$ for the complex numbers $z$ and $w$.

**Solution 3a.** Let $z = a + bi$ and $w = c + di$. Then,

$$|z \cdot w| = |(a + bi) \cdot (c + di)| = |(ac - bd) + i(ad + bc)|$$

$$\sqrt{(ac - bd)^2 + (ad + bc)^2} =$$

$$\sqrt{a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2} =$$

$$\sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}.$$

If we instead compute $|z| \cdot |w|$, then we have

$$|z| \cdot |w| = |a + bi| \cdot |c + di| =$$

$$\sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2} = \sqrt{(a^2 + b^2)(c^2 + d^2)} =$$

$$\sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}.$$

Notice that the above results for $|z \cdot w|$ and $|z| \cdot |w|$ are equal and thus

$$|z \cdot w| = |z| \cdot |w|.$$

□

**Solution 3b.** Let $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$. Then,

$$|z \cdot w| = \left|r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2}\right| = \left|(r_1 r_2) e^{i(\theta_1 + \theta_2)}\right| = r_1 r_2.$$

If we instead compute $|z| \cdot |w|$, we then have

$$|z| \cdot |w| = \left|r_1 e^{i\theta_1}\right| \cdot \left|r_2 e^{i\theta_2}\right| = r_1 \cdot r_2.$$

Notice that the above results for $|z \cdot w|$ and $|z| \cdot |w|$ are equal and thus

$$|z \cdot w| = |z| \cdot |w|.$$

□

---

**Question 4.** Below, we will prove a remarkable fact about real polynomials using complex numbers. For the below, let $z = a + bi$ and $w = c + di$ be complex numbers.

(a) Show that $\overline{z + w} = \overline{z} + \overline{w}$.

(b) Show that $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$.

(c) Use (b) to show that $\overline{z^n} = (\overline{z})^n$ for any natural number $n \in \mathbb{N}$.

(d) Consider the following polynomial $p(z)$ with *real coefficients*:

$$p(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0,$$

where each $\alpha_i$ is a real number. Show that if a complex number $w$ is a root to the above polynomial with real coefficients, then its conjugate $\overline{w}$ is also a root to the same polynomial. That is, use (a) - (c) to show that if $p(w) = 0$, then $p(\overline{w}) = 0$.

**Solution 4a.** Beginning with the left-hand side, we have

$$\overline{z+w} = \overline{(a+bi)+(c+di)} = \overline{(a+c)+i(b+d)} =$$

$$(a+c)-i(b+d) = (a-bi)+(c-di) = \overline{z}+\overline{w},$$

as desired.

$\square$

**Solution 4b.** Expanding the $\overline{z \cdot w}$, we have

$$\overline{z \cdot w} = \overline{(a+bi)(c+di)} = \overline{(ac-bd)+i(ad+bc)} = (ac-bd)-i(ad+bc).$$

Expanding $\overline{z} \cdot \overline{w}$, we have

$$\overline{z} \cdot \overline{w} = \overline{(a+bi)} \cdot \overline{(c+di)} = (a-bi)(c-di) =$$

$$ac - adi - bci - bd = (ac-bd)-i(ad+bc).$$

Since the above results for $\overline{z \cdot w}$ and $\overline{z} \cdot \overline{w}$ are equal, then $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$, as desired.

$\square$

**Solution 4c.** In (b), we proved that $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$. Letting $z = w$, we have

$$\overline{z^2} = \overline{z \cdot z} = \overline{z} \cdot \overline{z} = (\overline{z})^2 .$$

Continuing in this fashion with $n$ copies of $z$, we have

$$\overline{z^n} = \overline{z \cdot z \cdot z \cdots z} = \overline{z} \cdot \overline{z} \cdots \overline{z} = (\overline{z})^n ,$$

as desired.

$\square$

**Solution 4d.** Assume $w$ is a root to $p(z)$. Thus, $p(w) = 0$ and so

$$p(w) = \alpha_n w^n + \alpha_{n-1} w^{n-1} + \cdots + \alpha_1 w + \alpha_0 = 0.$$

If we take the conjugate of the above equation, we get that

$$\overline{\alpha_n w^n + \alpha_{n-1} w^{n-1} + \cdots + \alpha_1 w + \alpha_0} = \overline{0} = 0.$$

Using the above properties of the conjugate on the left-hand side of the equation, we have that

$$\begin{aligned}
\overline{\alpha_n w^n + \alpha_{n-1} w^{n-1} + \cdots + \alpha_1 w + \alpha_0} &= 0 \\
\overline{\alpha_n w^n} + \overline{\alpha_{n-1} w^{n-1}} + \cdots + \overline{\alpha_1 w} + \overline{\alpha_0} &= 0 \\
\overline{\alpha_n}\,\overline{w^n} + \overline{\alpha_{n-1}}\,\overline{w^{n-1}} + \cdots + \overline{\alpha_1 w} + \overline{\alpha}_0 &= 0 \\
\overline{\alpha_n}\,\overline{w}^n + \overline{\alpha_{n-1}}\,\overline{w}^{n-1} + \cdots + \overline{\alpha_1}\,\overline{w} + \overline{\alpha}_0 &= 0.
\end{aligned}$$

Since the $\alpha_i$ are real, then $\overline{\alpha_i} = \alpha_i$ for all $i$. Thus, the above is equivalent to

$$\alpha_n\,\overline{w}^n + \alpha_{n-1}\,\overline{w}^{n-1} + \cdots + \alpha_1\,\overline{w} + \alpha_0 = 0.$$

This is simply the polynomial $p(z)$ with an input of $\overline{w}$. Thus, $p(\overline{w}) = 0$ and $\overline{w}$ is a root.

$\square$