Transition to Mathematical Proofs
Chapter 4 - Sets of Real Numbers Assignment Solutions

Question 1. Let $a, b \in \mathbb{Z}$. Show that $4 \mid a^2 - b^2$ if and only if a and b are of the same parity.

Discussion 1. This is a biconditional statement $p \Leftrightarrow q$ with p being "$4 \mid a^2 - b^2$" and q being "a and b have the same parity". Thus, we have two statements to prove:

- $p \Rightarrow q$: "If $4 \mid a^2 - b^2$, then a and b have the same parity." For this statement, will will instead prove the contrapositive statement $\neg q \Rightarrow \neg q$ given by "If a and b have opposite parity, then $4 \nmid a^2 - b^2$."
- $q \Rightarrow p$: "If a and b have the same parity, then $4 \mid a^2 - b^2$. This can be proven directly.

`What we know:`

- For $\neg q \Rightarrow \neg p$, we know that a and b have opposite parity. Thus, a is odd and b is even or a is even and b is odd. Since the statement is symmetric in a and b, we only need a proof for one of the cases. So, we will assume that a is odd and b is even.
- For $q \Rightarrow p$, we know that a and b have the same parity. Thus, they are both even or both odd. We will proceed with a proof by cases.

`What we want:`

- For $\neg q \Rightarrow \neg p$, we wish to conclude that $4 \nmid a^2 - b^2$. Thus, if we can write $a^2 - b^2$ as one of the three possibilities: $4x + 1, 4x + 2$, or $4x + 3$ for some $x \in \mathbb{Z}$, then it is not divisible by 4.
- For $q \Rightarrow p$, we wish to conclude that $4 \mid a^2 - b^2$. Thus, we want to write $a^2 - b^2$ as $4x$ for some $x \in \mathbb{Z}$.

Proof 1. To prove this biconditional statement, we will prove the two conditional statements "If $4 \mid a^2 - b^2$, then a and b have the same parity" and "If a and b have the same parity, then $4 \mid a^2 - b^2$."

For the first conditional statement, we will instead prove the contrapositive: "If a and b have opposite parity, then $4 \nmid a^2 = b^2$." So, if a and b have opposite parity, then a is odd and b is even or a is even and b is odd. By the symmetry of the statement we wish to prove, we can consider only the first case; the second case will be almost identical. Thus, we assume that a is odd and b is even. So, there exist integers $k_1, k_2 \in \mathbb{Z}$ such that $a = 2k_1 + 1$ and $b = 2k_2$. Thus,

$$a^2 - b^2 = (2k_1 + 1)^2 - (2k_2)^2 = (4k_1^2 + 4k_1 + 1) - 4k_2^2 =$$

$$4k_1^2 - 4k_2^2 + 4k_1 + 1 = 2(2k_1^2 - 2k_2^2 + 2k_1) + 1.$$

Since $k_1, k_2 \in \mathbb{Z}$, then $2k_1^2 - 2k_2^2 + 2k_1 \in \mathbb{Z}$. So, since $a^2 - b^2 = 2(2k_1^2 - 2k_2^2 + 2k_1) + 1$, then $4 \nmid a^2 - b^2$. Thus, the contrapositive is true and the original statement "If $4 \mid a^2 - b^2$, then a and b have the same parity" is a true statement.

For the second conditional statement, we assume that a and b have the same parity. Thus, a and b are even or a and b are odd, giving us two cases. In the first case, a and b are even, so there exist integers $k_1, k_2 \in \mathbb{Z}$ such that $a = 2k_1$ and $b = 2k_2$. Thus,

$$a^2 - b^2 = (2k_1)^2 - (2k_2)^2 = 4k_1^2 - 4k_2^2 = 4(k_1^2 - k_2^2).$$

Since $k_1, k_2 \in \mathbb{Z}$, then $k_1^2 - k_2^2 \in \mathbb{Z}$. Since $a^2 - b^2 = 4(k_1^2 - k_2^2)$, then $4 \mid a^2 - b^2$. For the second case, assume that both a and b are odd. Thus, there exist integers $k_1, k_2 \in \mathbb{Z}$ such that $a = 2k_1 + 1$ and $b = 2k_2 + 1$. Thus,

$$a^2 - b^2 = (2k_1 + 1)^2 + (2k_2 + 1)^2 =$$

$$(4k_1^2 + 4k_1 + 1) - (4k_2^2 + 4k_2 + 1) = 4(k_1^2 - k_2^2 + k_1 - k_2).$$

Since $k_1, k_2 \in \mathbb{Z}$, then $k_1^2 - k_2^2 + k_1 - k_2 \in \mathbb{Z}$. Since $a^2 - b^2 = 4(k_1^2 - k_2^2 + k_1 - k_2)$, then $4 \mid a^2 - b^2$. Since, in either case, $4 \mid a^2 = b^2$, then our conditional statement is true.

Since we have proven both conditional statements, our biconditional statement "$4 \mid a^2 - b^2$ if and only if a and b have the same parity" is true.

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Question 2.

(a) Let $a \in \mathbb{Z}$. Show that $3 \mid a$ if and only if $3 \mid a^2$.

(b) Use (a) to show that $\sqrt{3}$ is irrational.

Discussion 2a. This is a biconditional statement $p \Leftrightarrow q$ with p being "$3 \mid a$" and q being "$3 \mid a^2$." Thus, we will break this up into its two conditional statements:

- $p \Rightarrow q$: "If $3 \mid a$, then $3 \mid a^2$." This will be a direct proof.
- $q \Rightarrow p$: "If $3 \mid a^2$, then $3 \mid a$." We will instead prove the contrapositive statement $\neg p \Rightarrow \neg q$ given by "If $3 \nmid a$, then $3 \nmid a^2$."

`What we know:`

- For $p \Rightarrow q$, we will assume that $3 \mid a$. Thus, there exists a $k \in \mathbb{Z}$ such that $a = 3k$.
- For $\neg p \Rightarrow \neg q$, we will assume that $3 \nmid a$. Thus, there exists a $k \in \mathbb{Z}$ such that $a = 3k + 1$ or $a = 3k + 2$. This will give us two cases to consider.

`What we want:`

- For $p \Rightarrow q$, we will need to conclude that $3 \mid a^2$. Thus, we will want to show that $a^2 = 3x$ for some $x \in \mathbb{Z}$.
- For $\neg p \Rightarrow \neg q$, we will need to conclude that $3 \nmid a^2$. Thus, we will wan to show that, for some $x \in \mathbb{Z}$, $a^2 = 3x + 1$ or $a^2 = 3x + 2$.

Proof 2a. To prove this biconditional statement, we will prove the two conditional statements: "If $3 \mid a$, then $3 \mid a^2$" and "If $3 \mid a^2$, then $3 \mid a$."

To prove the first conditional statement, we assume that $3 \mid a$. Thus, $a = 3k$ for some $k \in \mathbb{Z}$. Thus,

$$a^2 = (3k)^2 = 9k^2 = 3(3k^2).$$

Since $k \in \mathbb{Z}$, then $3k^2 \in \mathbb{Z}$. Since $a^2 = 3(3k^2)$, then we can conclude that $3 \mid a^2$, as desired.

To prove the second conditional statement, we will instead prove its contrapositive: "If $3 \nmid a$, then $3 \nmid a^2$." Since $3 \nmid a$, then for some $k \in \mathbb{Z}$, we can a^2 as $a^2 = 3k + 1$ or $a^2 = 3k + 2$, giving us two cases. In the first case, $a = 3k + 1$ and thus

$$a^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1.$$

Since $k \in \mathbb{Z}$, then $3k^2 + 2k \in \mathbb{Z}$. Since $a^2 = 3(3k^2 + 2k) + 1$, then $3 \nmid a^2$, as desired. For the second case, $a = 3k + 2$. Thus,

$$a^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1.$$

Since $k \in \mathbb{Z}$, then $3k^2 + 4k + 1 \in \mathbb{Z}$. Since $a^2 = 3(3k^2 + 4k + 1) + 1$, we can conclude that $3 \nmid a^2$, as desired. In either case, we conclude that $3 \nmid a^2$. So, the contrapositive statement is true and, thus, our original conditional statement "If $3 \mid a^2$,, then $3 \mid a$" is also true.

Since we have proven both conditional statements, the biconditional statement " $3 \mid a$ if and only if $3 \mid a^2$" is true.

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Discussion 2b. This proof will proceed in a similar fashion to the proof that $\sqrt{2}$ is irrational. Thus, this will be a proof by contradiction, where we assume that $\sqrt{3}$ is rational and can thus be written as $\frac{p}{q}$ in lowest term. Then, we will ultimately arrive at a contradiction by showing that both p and q are divisible by 3. We will use one direction of the above biconditional twice throughout our proof: "If $3 \mid a^2$, then $3 \mid a$."

Proof 2b. Assume, to the contrary, that $\sqrt{3}$ is rational. Thus, we can write

$$\sqrt{3} = \frac{p}{q},$$

with $p, q \in \mathbb{Z}$, $q \neq 0$, and p and q have no common divisors. Squaring both sides of our equation, we obtain

$$3 = \frac{p^2}{q^2}$$

and, cross-multiplying, we obtain $3q^2 = p^2$. Thus, $3 \mid p^2$. Our previous proof shows us that if $3 \mid p^2$, then $3 \mid p$. Thus, we can write $p = 3r$ for some $r \in \mathbb{Z}$. Substituting, we obtain

$$3q^2 = (3r)^2 = 9r^2,$$

which is equivalent to $q^2 = 3r^2$. Thus, $3 \mid q^2$ and, by our previous proof, $3 \mid q$. Thus, $3 \mid p$ and $3 \mid q$, contradicting the fact that p and q have no common divisors. Thus, our initial assumption that $\sqrt{3}$ is rational is false and thus $\sqrt{3}$ is irrationa.

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Question 3. Let $a, b \in \mathbb{R}$. Show that if $a + b$ is rational, then a is irrational or b is rational.

Discussion 3. Our statement is the conditional statement $p \Rightarrow q$ with p being "$a + b$ is rational" and q being "a is irrational or b is rational." We will instead prove the contrapositive statement $\neg q \Rightarrow \neg p$. Notice that, to compute $\neg q$, we must use DeMorgan's Logic Law, which gives us that $\neg q$ is "a is rational and b is irrational." Thus, we will prove the contrapositive $\neg q \Rightarrow \neg p$ given by "If a is rational and b is irrational, then $a + b$ is irrational.

`What we know:` a is rational and b is irrational.

`What we want:` $a + b$ is irrational. Since this is a negative statement, we will use proof by contradiction. Thus, we will assume that $a + b$ is rational and arrive at a contradiction.

Proof 3. To prove our statement, we will instead prove its contrapositive: "If a is rational and b is irrational, then $a + b$ is irrational." Assume, to the contrary, that $a + b$ is rational. Then, since a is rational, $-a$ is also rational. Since the sum of two rational numbers is rational, then $(a + b) - a = b$ is also rational. This contradicts, however, that b is irrational. Thus, our initial assumption that $a + b$ is rational is false, and thus $a + b$ is irrational. So, we proven the contrapositive to be true and thus the original statement "If $a + b$ is rational, then a is irrational or b is rational" is also true.

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