Transition to Mathematical Proofs
Chapter 1 - Logic Assignment Solutions

Question 1. Let $m \neq 0$ and b be real numbers. Show that there exists a unique x such that $mx + b = 0$.

Discussion 1.

`What we want:` We want to find an x that satisfies $mx + b = 0$. Solving for x, we can see that $x = \frac{-b}{m}$ would be a good choice.

`What we'll do:` We will verify that $x = \frac{-b}{m}$ will work by plugging it into the equation. To show uniqueness, we will assume that $mx + b = 0$ and $my + b = 0$ and conclude that $x = y$ using basic algebra.

Proof 1. Consider $x = -\frac{b}{m}$. Since $m \neq 0$, x is indeed a real number. Notice that

$$mx + b = m\left(\frac{-b}{m}\right) + b = -b + b = 0,$$

and thus a solution exists.

To show uniqueness, assume that there exist real numbers x and y such that $mx + b = 0$ and $my + b = 0$; we will show that $x = y$. So, $mx + b = 0 = my + b$ and thus $mx + b = my + b$. Subtract a b from both sides yields $mx = my$. Since $m \neq 0$, we can divide by m to obtain that $x = y$. Thus, a unique solution to $mx + b = 0$ exists.

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Question 2. Prove the following biconditional statement.

Let x be a real number. $-1 \leq x \leq 1$ if and only if $x^2 \leq 1$.

In proving this, it may be helpful to note that $-1 \leq x \leq 1$ is equivalent to $-1 \leq x$ and $x \leq 1$.

Discussion 2.

The biconditional statement "$-1 \leq x \leq 1$ if and only if $x^2 \leq 1$" can be thought of as $p \Leftrightarrow q$ with p being the statement "$-1 \leq x \leq 1$" and q being the statement "$x^2 \leq 1$". Thus, we we will prove the following two conditional statements:

- $p \Rightarrow q$: If $-1 \leq x \leq 1$, then $x^2 \leq 1$.
- $q \Rightarrow p$: If $x^2 \leq 1$, then $-1 \leq x \leq 1$.

`What we'll do:` For $p \Rightarrow q$, we will perform a direct proof but will break up the hypothesis $-1 \leq x \leq 1$ into two cases: $-1 \leq x < 0$ or $0 \leq x \leq 1$. In both cases, we will use basic algebra to arrive at the conclusion $x^2 \leq 1$.

For $q \Rightarrow p$, since the hypothesis deals with x^2 instead of x, we will instead prove its contrapositive: $\neg p \Rightarrow \neg q$. Since p is $-1 \leq x \leq 1$, then its negation $\neg p$, using DeMorgan's Logic Law, is "$x < -1$ or $x > 1$". Since q is "$x^2 \leq 1$", its negation is $\neg q$ is "$x^2 > 1$". Thus, the contrapositive $\neg p \Rightarrow \neg q$ that we will prove is the statement "If $x < -1$ or $x > 1$, then $x^2 > 1$". This will be proven with cases: $x < -1$ or $x > 1$.

Proof 2. To prove our biconditional statement, we will prove its two corresponding conditional statements.

First, we prove "If $-1 \leq x \leq 1$, then $x^2 \leq 1$." We split up our proof into two cases: $-1 \leq x < 0$ or $0 \leq x \leq 1$. In our first case, if $-1 \leq x < 0$, then we can multiply through by -1 to obtain $0 < -x \leq 1$. Since both $-x$ and 1 are positive, we can multiply the inequality $-x \leq 1$ by itself to obtain $(-x) \cdot (-x) \leq 1 \cdot 1$, which is equivalent to $x^2 \leq 1$, as desired. In the second case, since $0 \leq x \leq 1$, then x and 1 are non-negative, and we can multiply the inequality $x \leq 1$ by itself to obtain $x \cdot x \leq 1 \cdot 1$, which is equivalent to $x^2 \leq 1$, as desired. In either case, $x^2 \leq 1$; thus, we have proven that if $-1 \leq x \leq 1$, then $x^2 \leq 1$.

Next, we prove the conditional statement "If $x^2 \leq 1$, then $-1 \leq x \leq 1$" by instead proving its contrapositive: "If $x < -1$ or $x > 1$, then $x^2 > 1$." We will use two cases to prove this statement. In our first case, $x < -1$. Multiplying through by -1 yields $1 < -x$. Since both 1 and $-x$ are positive, we can multiply the inequality $1 < -x$ by itself to obtain $1 \cdot 1 < (-x) \cdot (-x)$, which is equivalent to $1 < x^2$. In the second case, $x > 1$. Since both x and 1 are positive, we can multiply the inequality by itself to obtain $1 \cdot 1 < x \cdot x$, which is equivalent to $1 < x^2$. In either case, $1 < x^2$; thus, we have proven that if $x < -1$ or $x > 1$, then $1 < x^2$, which is the contrapositive of the desired statement: "If $x^2 < 1$, then $-1 \leq x \leq 1$."

Having proven both conditional statements, the entire biconditional statement "$-1 \leq x \leq 1$ if and only if $x^2 \leq 1$" is true.

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Question 3. Two whole numbers are said to *have the same parity* if they are both even or both odd. Prove the following biconditional statement:

> Let m and n be whole numbers. m and n have the same parity if and only if $m + n$ is even.

Discussion 3. We are asked to prove the biconditional statement "m and n have the same parity if and only if $m + n$ is even." We can write this as $p \Leftrightarrow q$ where p is "m and n have the same parity," and q is "$m + n$ is even". Thus, we will prove the following two conditional statements:

- $p \Rightarrow q$: If m and n have the same parity, then $m + n$ is even.
- $q \Rightarrow p$: If $m + n$ is even, then m and n have the same parity.

`What we'll do:` For $p \Rightarrow q$, we will perform a direct proof, but we will split up the hypothesis "m and n have the same parity" into two cases: "m and n are even" or "m and n are odd". In both cases, we will conclude that $m + n$ is even by writing it as 2(whole number).

For $q \Rightarrow p$, since the hypothesis "$m + n$ is even" is difficult to work with, we will instead prove its contrapositive $\neg p \Rightarrow \neg q$. Since p is "m and n have the same parity", its negation $\neg p$ is given by "m and n have opposite parity". Since q is "$m + n$ is even", then its negation $\neg q$ is given by "$m + n$ is odd". Thus, we will will prove the contrapositive statement $\neg p \Rightarrow \neg q$, which is "If m and n have opposite parity, then $m + n$ is odd." We will do this by assuming that one of m and n is even and the other is odd. Since this statement is symmetric with respect to m and n, we can assume that m is even and n is odd. We will use this to show that $m + n$ is odd by writing it as 2(whole number) + 1.

Proof 3. To prove our biconditional statement, we will prove its two corresponding conditional statements.

First, we prove "If m and n have the same parity, then $m+n$ is even." We consider the two cases: m and n are both even or m and n are both odd. In the first case, m and n are both even, thus, we can write them as $m = 2k_1$ and $n = 2k_2$ for some whole numbers k_1 and k_2. Thus,

$$m + n = 2k_1 + 2k_2 = 2(k_1 + k_2).$$

Since k_1 and k_2 are whole numbers, their sum $k_1 + k_2$ is a whole number and thus $m + n$ is even. For the second case, assume that m and n are both odd. Thus, we can write them as $m = 2k_1 + 1$ and $n = 2k_2 + 1$ for some whole numbers k_1 and k_2. Thus,

$$m + n = (2k_1 + 1) + (2k_2 + 1) = 2k_1 + 2k_2 + 2 = 2(k_1 + k_1 + 1).$$

Since k_1 and k_2 are whole numbers, then $k_1 + k_2 + 1$ is also a whole number. Thus, $m+n$ is even. In either case, $m+n$ is an even number and our conditional statement is true.

Next, we prove the conditional statement "If $m + n$ are even, then m and n have the same parity" by proving instead its contrapositive: "If m and n are of opposite parity, then $m+n$ is odd." If m and n are of opposite parity, then one is even and one is odd. We can assume that m is even and n is odd; the proof of the case where m is odd and n is even is almost identical. Since m is even and n is odd, we can write them as $m = 2k_1$ and $n = 2k_2 + 1$ for some whole numbers k_1 and k_2. Thus,

$$m + n = 2k_1 + 2k_2 + 1 = 2(k_1 + k_2) + 1.$$

Since k_1 and k_2 are whole numbers $k_1 + k_2$ is also a whole number and thus $m + n$ is an odd number. Thus, we have proven the contrapositive, and thus the original conditional statement "If $m + n$ are even, then m and n have the same parity" is true.

Since we proved both conditional statements, our original biconditional statement "m and n have the same parity if and only if $m + n$ is even" is true.

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Question 4. Use *proof by contrapositive* to prove the following conditional statement.

> Let m and n be whole numbers. If $m \cdot n$ is odd, then m and n are both odd.

Discussion 4. We wish to show the conditional statement "If $m \cdot n$ is odd, then m and n are both odd." We can think of this statement as $p \Rightarrow q$ with p being "$m \cdot n$ is odd", and q being "m and n are odd". Since the hypothesis p involved information about $m \cdot n$ (as opposed to m and n individually), we will instead prove the contrapositive $\neg q \Rightarrow \neg p$. Since q is the statement "m and n are both odd", we can negate this conjunction using DeMorgan's Logic Laws to obtain the negation $\neg q$ as "m is even or n is even". Since p is "$m \cdot n$ is odd", its negation $\neg p$ is given by "$m \cdot n$ is even". Thus, we will prove the contrapositive $\neg q \Rightarrow \neg p$ given by "If m is even or n is even, then $m \cdot n$ is even."

`What we know:` m is even or n is even. Thus, we will use cases. However, since the statement is symmetric in m and n, we only need to consider the case when one of them (say m) is even. Thus, we can write $m = 2k$ for some whole number k.

`What we want:` $m \cdot n$ is even. Thus, we must write $m \cdot n = 2(\text{whole number})$.

Proof 4. To prove our statement, we will instead prove its contrapositive: "If m is even or n is even, them $m \cdot n$ is even."

We have two cases: m is even or n is even. We will only consider the case where m is even since the case where n is even is almost identical. Since m is even, it can be written as $m = 2k$ where k is a whole number. Thus,

$$m \cdot n = 2k \cdot n = 2(kn).$$

Since k and n are whole numbers, kn is also a whole number and thus $m \cdot n$ is even.

Having proven the contrapositive, the original conditional statement "If $m \cdot n$ is odd, then m and n are both odd" is true.

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Question 5. We will investigate the following statement:

> Every odd whole number can be written as the difference of two perfect squares.

(a) For the odd whole numbers $n = -3, -1, 1, 3, 5, 7, 9$, write n as the difference of two perfect squares.

(b) Use any pattern that you found in (a) to help you write a proof of our statement.

Solution 5a. We write the following odd whole numbers as the difference of two perfect squares:

$$\begin{aligned} -3 &= 1^2 - 2^2 \\ -1 &= 0^2 - 1^2 \\ 1 &= 1^2 - 0^2 \\ 3 &= 2^2 - 1^2 \\ 5 &= 3^2 - 2^2 \\ 7 &= 4^2 - 3^2 \\ 9 &= 5^2 - 4^2 \end{aligned}$$

Discussion 5b.

`What we know:` From our previous computations, it seems that a very clear pattern has emerged in terms of writing an odd number as a difference of squares. In particular, the two numbers that we square are only 1 number apart.

`What we'll do:` Since n is an odd numbers, we can write it as $n = 2k + 1$ for some whole number k. Looking at our pattern, it seems that $n = 2k + 1$ is the difference of $(k+1)^2$ and k^2. We will verify this algebraically.

Proof 5b. Let n be an odd whole number. We can then write it as $n = 2k + 1$ for some whole number k. Notice that

$$(k+1)^2 - k^2 = (k^2 + 2k + 1) - k^2 = 2k + 1 = n,$$

and thus we have written the odd number n as the difference of the two squares $(k+1)^2$ and k^2.

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