Transition to Mathematical Proofs
Chapter 2 - Set Theory Assignment Solutions

Question 1. Let S and T be sets. Then

$$\overline{S \cap T} = \overline{S} \cup \overline{T}.$$

Discussion 1.

`What we want:` $\overline{S \cap T} = \overline{S} \cup \overline{T}$. Thus we will show the following two subset inclusions: $\overline{S \cap T} \subset \overline{S} \cup \overline{T}$ and $\overline{S} \cup \overline{T} \subset \overline{S \cap T}$.

`What we'll do:` We will be using DeMorgan's Logic Law, which states that $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

Proof 1. To prove that $\overline{S \cap T} = \overline{S} \cup \overline{T}$, we will prove the following two subset inclusions: $\overline{S \cap T} \subset \overline{S} \cup \overline{T}$ and $\overline{S} \cup \overline{T} \subset \overline{S \cap T}$.

For the first subset inclusion, let $x \in \overline{S \cap T}$. Thus, $x \notin S \cap T$ and it is not true that $x \in S$ and $x \in T$. By DeMorgan's Logic Law, this is equivalent to $x \notin S$ or $x \notin T$. In other words, $x \in \overline{S}$ or $x \in \overline{T}$. Thus, $x \in \overline{S} \cup \overline{T}$. So, $\overline{S \cap T} \subset \overline{S} \cup \overline{T}$.

For the second subset inclusion, let $x \in \overline{S} \cup \overline{T}$. Thus, $x \in \overline{S}$ or $x \in \overline{T}$, and so $x \notin S$ or $x \notin T$. By DeMorgan's Logic Law, this is equivalent to it not being true that $x \in S$ and $x \in T$. Therefore, it is not true that $x \in S \cap T$ and so $x \notin S \cap T$. Thus, $x \in \overline{S \cap T}$. So, $\overline{S} \cup \overline{T} \subset \overline{S \cap T}$.

Since we have proven both subset inclusions, we may conclude that $\overline{S \cap T} = \overline{S} \cup \overline{T}$.

□

Question 2. Let S, T, and R be sets. Then,

$$S \cap (T \cup R) = (S \cap T) \cup (S \cap R).$$

Discussion 2.

`What we want:` $S \cap (T \cup R) = (S \cap T) \cup (S \cap R)$. Thus, we will show the following two subset inclusions: $S \cap (T \cup R) \subset (S \cap T) \cup (S \cap R)$ and $(S \cap T) \cup (S \cap R) \subset S \cap (T \cup R)$.

`What we'll do:` For the first subset inclusion $S \cap (T \cup R) \subset (S \cap T) \cup (S \cap R)$, we will let $x \in S \cap (T \cup R)$. Our job will be to show that $x \in (S \cap T) \cup (S \cap R)$. Since $x \in S \cap (T \cup R)$, we know that $x \in S$ and also that $x \in T \cup R$. Since $x \in T \cup R$, we have that $x \in T$ or $x \in R$. Using these two cases, we will conclude that $x \in (S \cap T) \cup (S \cap R)$.

For the second subset inclusion $(S \cap T) \cup (S \cap R) \subset S \cap (T \cup R)$, we will let $x \in (S \cap T) \cup (S \cap R)$ and will conclude that $x \in S \cap (T \cup R)$. Since $x \in (S \cap T) \cup (S \cap R)$, we know that $x \in S \cap T$ or $x \in S \cap R$, leaving us with two cases. In each of these cases, we will conclude that $x \in S \cap (T \cup R)$.

Proof 2. We will prove that $S \cap (T \cup R) = (S \cap T) \cup (S \cap R)$ by proving the two subset inclusions: $S \cap (T \cup R) \subset (S \cap T) \cup (S \cap R)$ and $(S \cap T) \cup (S \cap R) \subset S \cap (T \cup R)$.

For the first inclusion, let $x \in S \cap (T \cup R)$. Then, $x \in S$ and $x \in T \cup R$. Since $x \in T \cup R$, $x \in T$ or $x \in R$, giving us two cases. In the first case,

$x \in T$. Since $x \in S$ and $x \in T$, then $x \in S \cap T \subset (S \cap T) \cup (S \cap R)$, and thus $x \in (S \cap T) \cup (S \cap R)$. In the second case, $x \in R$. Since $x \in S$ and $x \in R$, then $x \in S \cap R \subset (S \cap T) \cup (S \cap R)$. In either case, we conclude that $x \in (S \cap T) \cup (S \cap R)$ and thus $S \cap (T \cup R) \subset (S \cap T) \cup (S \cap R)$.

For the second inclusion, let $x \in (S \cap T) \cup (S \cap R)$. Thus, $x \in S \cap T$ or $x \in S \cap R$, giving us two cases. In the first case, $x \in S \cap T$. Then $x \in S$ and $x \in T$. Since $x \in T$, then $x \in T \subset T \cup R$ and so $x \in T \cup R$. Thus, $x \in S$ and $x \in T \cup R$ and therefore $x \in S \cap (T \cup R)$. In the second case, $x \in S \cap R$ and thus $x \in S$ and $x \in R$. Since $x \in R$, then $x \in R \subset T \cup R$ and so $x \in T \cup R$. Thus, $x \in S$ and $x \in (T \cup R)$, and so $x \in S \cap (T \cup R)$. In either case, we conclude that $x \in S \cap (T \cup R)$ and therefore $(S \cap T) \cup (S \cap R) \subset S \cap (T \cup R)$.

Since we have proven both subset inclusions, we may conclude that $S \cap (T \cup R) = (S \cap T) \cup (S \cap R)$.

□

Question 3. Let A, B, C, and D be sets. Show that if $A \subset B$ and $C \subset D$, then $A \times C \subset B \times D$.

Discussion 3.

`What we know:`

- $A \subset B$. Thus, whenever we know that $x \in A$, we can conclude that $x \in B$.
- $C \subset D$. Thus, whenever we know that $x \in C$, we can conclude that $x \in D$.

`What we want:` $A \times C \subset B \times D$. We will take an arbitrary element in $A \times C$ and show that it is in $B \times D$.

`What we'll do:` In choosing an arbitrary element in the product $A \times C$, we may write it as a 2-tuple $(a, c) \in A \times C$ where $a \in A$ and $c \in C$. Our job is to show that $(a, c) \in B \times D$ by showing that $a \in B$ and $c \in D$.

Proof 3. Let $(a, c) \in A \times C$. We will show that $(a, c) \in B \times D$. Since $(a, c) \in A \times C$, then $a \in A$ and $c \in C$. Since $a \in A$ and $A \subset B$, then $a \in B$. Since $c \in C$ and $C \subset D$, then $c \in D$. Since $a \in B$ and $c \in D$, then $(a, c) \in B \times D$, as desired. So, $A \times C \subset B \times D$.

□

Question 4. Let A, B, and C be sets. Show that

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

Discussion 4.

`What we want:` $A \times (B \cap C) = (A \times B) \cap (A \times C)$. Thus, we will show the following two subset inclusions: $A \times (B \cap C) \subset (A \times B) \cap (A \times C)$ and $(A \times B) \cap (A \times C) \subset A \times (B \cap C)$.

`What we'll do:` For the first subset inclusion $A \times (B \cap C) \subset (A \times B) \cap (A \times C)$, we will take an element in $A \times (B \cap C)$ and eventually conclude that this element is also in $(A \times B) \cap (A \times C)$. An arbitrary element in $A \times (B \cap C)$ looks like

(x, y), where $x \in A$ and $y \in B \cap C$. We will use these facts to show that $(x, y) \in (A \times B) \cap (A \times C)$.

For the second inclusion, $(A \times B) \cap (A \times C) \subset A \times (B \cap C)$, we will take a general element in $(A \times B) \cap (A \times C)$ and eventually conclude that this element is also in $A \times (B \cap C)$. A general element in $(A \times B) \cap (A \times C)$ looks like a 2-tuple (x, y). Since $(x, y) \in (A \times B) \cap (A \times C)$, we know that $(x, y) \in A \times B$ and that $(x, y) \in A \times C$. Using this, we will conclude that $(x, y) \in A \times (B \cap C)$.

Proof 4. To show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$, we will show the following two subset inclusions: $A \times (B \cap C) \subset (A \times B) \cap (A \times C)$ and $(A \times B) \cap (A \times C) \subset A \times (B \cap C)$.

For the first subset inclusion, let $(x, y) \in A \times (B \cap C)$. Thus, $x \in A$ and $y \in B \cap C$. Since $x \in B \cap C$, then $y \in B$ and $y \in C$. Thus, $x \in A$ and $y \in B$ and thus $(x, y) \in A \times B$. Furthermore, $x \in A$ and $y \in C$ and thus $(x, y) \in A \times C$. Since (x, y) is an element of both $A \times B$ and $A \times C$, we know that $(x, y) \in (A \times B) \cap (A \times C)$. Thus, $A \times (B \cap C) \subset (A \times B) \cap (A \times C)$.

For the second subset inclusion, let $(x, y) \in (A \times B) \cap (A \times C)$. Thus, $(x, y) \in A \times B$ and $(x, y) \in A \times C$. Since $(x, y) \in A \times B$, then $x \in A$ and $y \in B$. Since $(x, y) \in A \times C$, then $x \in A$ and $y \in C$. Thus, $x \in A$. Furthermore, since $y \in B$ and $y \in C$, then $y \in B \cap C$. Thus, $(x, y) \in A \times (B \cap C)$. So, $(A \times B) \cap (A \times C) \subset A \times (B \cap C)$.

Having proven both subset inclusions above, we can now conclude the desired set equality: $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Question 5. Let A, B, and C be sets. Show that

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

Discussion 5.

`What we want:` $A \times (B \cup C) = (A \times B) \cup (A \times C)$. Thus, we will show the two following subset inclusions: $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$ and $(A \times B) \cup (A \times C) \subset A \times (B \cup C)$.

`What we'll do:` For the first subset inclusion $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$, we will take an arbitrary element in $A \times (B \cup C)$ and show that it is also an element of $(A \times B) \cup (A \times C)$. An arbitrary element in $A \times (B \cup C)$ looks like (x, y) with $x \in A$ and $y \in B \cup C$. Since $y \in B \cup C$, we will use two cases: $y \in B$ or $y \in C$. In both cases, we will conclude that $(x, y) \in (A \times B) \cup (A \times C)$.

For the second subset inclusion, we will take an arbitrary element of $(A \times B) \cup (A \times C)$ and eventually show that it is also in $A \times (B \cup C)$. A general element in $(A \times B) \cup (A \times C)$ is of the form (x, y). Since it is part of the union, we will have two cases: $(x, y) \in A \times B$ or $(x, y) \in A \times C$. In both cases, we will conclude that $(x, y) \in A \times (B \cup C)$.

Proof 5. We will show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$ by showing the following two subset inclusions: $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$ and $(A \times B) \cup (A \times C) \subset A \times (B \cup C)$.

For the first subset inclusion, let $(x, y) \in A \times (B \cup C)$. Thus, $x \in A$ and $y \in B \cup C$. Since $y \in B \cup C$, then $y \in B$ or $y \in C$, giving us two cases to consider. In the first case, $y \in B$. Since $x \in A$ and $y \in B$, then $(x, y) \in A \times B$. Thus, $(x, y) \in A \times B \subset (A \times B) \cup (A \times C)$, and so $(x, y) \in (A \times B) \cup (A \times C)$. In the second case, $y \in C$. Since $x \in A$ and $y \in C$, then $(x, y) \in A \times C$. Thus,

$(x, y) \in A \times C \subset (A \times B) \cup (A \times C)$, and so $(x, y) \in (A \times B) \cup (A \times C)$. In either of the two cases, $(x, y) \in (A \times B) \cup (A \times C)$. Thus, $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$.

For the second subset inclusion, let $(x, y) \in (A \times B) \cup (A \times C)$. Then, $(x, y) \in A \times B$ or $(x, y) \in A \times C$, giving us two cases. In the first case, $(x, y) \in A \times B$ and thus $x \in A$ and $y \in B$. Since $y \in B$, then $y \in B \cup C$. Since $x \in A$ and $y \in B \cup C$, then $(x, y) \in A \times (B \cup C)$. In the second case, $(x, y) \in A \times C$. Thus, $x \in A$ and $y \in C$. Since $y \in C$, then $y \in B \cup C$. So, since $x \in A$ and $y \in B \cup C$, then $(x, y) \in A \times (B \cup C)$. In either case, we have that $(x, y) \in A \times (B \cup C)$ and thus $(A \times B) \cup (A \times C) \subset A \times (B \cup C)$.

Since we have shown both subset inclusions, we can conclude that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

$\square$