# Transition to Mathematical Proofs
## Chapter 6 - Induction Assignment Solutions

**Question 1.** Let $r \neq 1$ be a real number. Use mathematical induction to show that

$$\sum_{j=0}^{n} r^j = \frac{1 - r^{n+1}}{1 - r}.$$

**Solution 1.** We will use mathematical induction to show that the statement $A(n)$ given by

$$\sum_{j=0}^{n} r^j = \frac{1 - r^{n+1}}{1 - r}$$

is true for all integers $n \geq 0$.

First, we verify that the base case $A(0)$ is true. $A(0)$ is the statement that

$$\sum_{j=0}^{0} r^j = \frac{1 - r^{0+1}}{1 - r}.$$

The left-hand side of the above equation is equivalent to $r^0 = 1$ and the right-hand side is equal to $\frac{1-r}{1-r} = 1$. Thus, since both sides of the equation are equal to 1, the equation in $A(0)$ is true.

For the inductive step, we assume that $A(k)$ is true for some $k \geq 0$. Thus, we assume that

$$\sum_{j=0}^{k} r^j = \frac{1 - r^{k+1}}{1 - r}$$

is true. We will use this to show that the statement $A(k+1)$ given by

$$\sum_{j=0}^{k+1} r^j = \frac{1 - r^{(k+1)+1}}{1 - r}$$

is true. Beginning with the left-hand side of the desired equation, we have

$$\sum_{j=0}^{k+1} r^j = \left(\sum_{j=0}^{k} r^j\right) + r^{k+1} = \frac{1 - r^{k+1}}{1 - r} + r^{k+1} =$$
$$\frac{1 - r^{k+1}}{1 - r} + \frac{r^{k+1}(1 - r)}{1 - r} = \frac{1 - r^{k+1}}{1 - r} + \frac{r^{k+1} - r^{k+2}}{1 - r} =$$
$$\frac{1 - r^{k+1} + r^{k+1} - r^{k+2}}{1 - r} = \frac{1 - r^{(k+1)+1}}{1 - r}.$$

Thus, $A(k+1)$ is true and the the inductive step is complete.

By induction, the statement $A(n)$ given by

$$\sum_{j=0}^{n} r^j = \frac{1 - r^{n+1}}{1 - r}$$

is true for all $n \geq 0$.

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**Question 2.** Consider the function $f(x) = \frac{1}{1-x}$.

(a) Compute the first several derivatives of $f$ and use them to conjecture a pattern for $f^{(n)}(x)$.

(b) Prove that your conjectured pattern for $f^{(n)}(x)$ is indeed true by using a proof by induction.

**Question 2a.** Computing the first few derivatives, we have

$$\begin{array}{rcl} f(x) & = & (1-x)^{-1} \\ f'(x) & = & (1-x)^{-2} \\ f''(x) & = & 2(1-x)^{-3} \\ f'''(x) & = & 6(1-x)^{-4} \\ f^{(4)}(x) & = & 24(1-x)^{-5} \end{array}$$

Thus, a reasonable pattern to conjecture is that

$$f^{(n)}(x) = n!(1-x)^{-(n+1)}$$

for all integers $n \geq 0$.

**Question 2b.** We will use mathematical induction to prove that the statement $A(n)$ given by

$$f^{(n)}(x) = n!(1-x)^{-(n+1)}$$

is true for all integers $n \geq 0$.

First, we verify the base case $A(0)$, which states that $f^{(0)}(x) = 0!(1-x)^{0-1}$, which is equivalent to $f(x) = (1-x)^{-1}$, a true statement.

Next, we perform the inductive step. Thus, we assume that $A(k)$ is true for some $k \geq 0$. So,

$$f^{(k)}(x) = k!(1-x)^{-(k+1)}.$$

We will use this to prove that $A(k+1)$ is true by showing that

$$f^{(k+1)}(x) = k!(1-x)^{-((k+1)+1)}.$$

Beginning with the left-hand side of the desired equation and using the fact that $(k+1)! = (k+1)k!$, we have

$$f^{k+1}(x) = \frac{d}{dx}k!(1-x)^{-(k+1)} =$$

$$k!\,\frac{d}{dx}(1-x)^{-(k+1)} =$$

$$k!\,(-1)(k+1)\cdot(-1)(1-x)^{-(k+1)-1} =$$

$$(k+1)!(1-x)^{-((k+1)+1)}.$$

Thus, the statement $A(k+1)$ is true.

By mathematical induction, we can conclude that $A(n)$ given by

$$f^{(n)}(x) = n!(1-x)^{-(n+1)}$$

is true for all $n \geq 0$.

□

**Question 3.** Let $x > -1$. Use mathematical induction to prove that

$$(1+x)^n \geq 1 + nx$$

for all integers $n \geq 1$.

**Solution 3.** We will use mathematical induction to prove that the statement $A(n)$ given by $(1+x)^n \geq 1+nx$ is true for all integers $n \geq 1$.

First, we verify the base case $A(1)$, which states that $(1+x)^1 \geq 1+x$, which is true because $1+x = 1+x$. Thus, $A(1)$ is true.

Next, we perform the inductive step. Thus, we will assume that $(1+x)^k \geq 1+kx$ for some $k \geq 1$. We will show that $A(k+1)$ is true by showing that

$$(1+x)^{k+1} \geq 1 + (k+1)x.$$

Beginning with the left-hand side of our desired inequality, and using the fact that $1+x > 0$ (since $x > -1$), we have

$$(1+x)^{k+1} = (1+x)(1+x)^k \geq (1+x)(1+kx) =$$

$$1 + kx + x + kx^2 = 1 + (k+1)x + kx^2.$$

Since $k \geq 1$ and $x^2 \geq 0$, then $kx^2 \geq 0$ and so $1+(k+1)x+kx^2 \geq 1+(k+1)x$. Putting these inequalities together, we have that

$$(1+x)^{k+1} \geq 1 + (k+1)x,$$

which is exactly the statement $A(k+1)$.

By induction, we can conclude that the statement $A(n)$ given by $(1+x)^n \geq 1+nx$ is true for all $n \geq 1$.

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