Transition to Mathematical Proofs
Chapter 3 - Functions Assignment Solutions

Question 1. Let $m \neq 0$ and b be real numbers and consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = mx + b$.

(a) Prove that f is a bijection.

(b) Since f is a bijection, it is invertible. Find its inverse f^{-1}, and show it is an inverse by demonstrating that

$$f^{-1}(f(x)) = x.$$

Discussion 1a. To show that $f(x) = mx + b$ is a bijection, we must show that it is both an injection and a surjection.

- To show that f is a surjection, we must find, for every $a \in \mathbb{R}$, an x such that $f(x) = mx + b = a$. Solving for x, we see that $x = \dfrac{a-b}{m}$ will work.
- To show that f is an injection, we will assume that $f(x) = f(y)$ and check algebraically that $x = y$.

Solution 1a. To show that $f(x) = mx + b$, we will show that it is both a surjection and an injection.

To show that $f(x)$ is a surjection, let $a \in \mathbb{R}$. We will find a pre-image for a. Notice that $x = \frac{a-b}{m} \in \mathbb{R}$ is indeed a real number since $m \neq 0$. Furthermore, x is a pre-image because

$$f\left(\frac{b-a}{m}\right) = m\left(\frac{b-a}{m}\right) + b = b - a + b = a.$$

Thus, every $a \in \mathbb{R}$ has a pre-image and f is surjective.

To show that $f(x)$ is an injection, assume that $f(x) = f(y)$. Then, $mx + b = my + b$. Subtracting b, we get $mx = my$ and dividing by $m \neq 0$, we get $x = y$, as desired. Thus, f is an injection.

Since f is both a surjection and an injection, it is a bijection.

□

Discussion 1b. To find the inverse, we can simply set $y = mx + b$ and solve for x. Doing so, we get that $x = \frac{y-b}{m}$ and thus it may be wise to defined $f^{-1}(x) = \frac{x-b}{m}$.

Proof 1b. For $f(x) = mx + b$, consider the function $f^{-1} : \mathbb{R} \to \mathbb{R}$ given by $f^{-1}(x) = \frac{x-b}{m}$. Notice that

$$f^{-1}(f(x)) = f^{-1}(mx + b) = \frac{(mx + b) - b}{m} = \frac{mx}{m} = x.$$

Thus, f^{-1} is indeed the inverse of f.

□

Question 2. Let $\gamma, \rho \in \mathbb{R}$ be real numbers such that $\gamma \cdot \rho \neq 1$. Let $\mathbb{R}-\{\gamma\}$ and $\mathbb{R}-\{-\rho\}$ be the set of all real numbers $\mathbb{R}$ except for γ and $-\rho$, respectively. Consider the function $f : \mathbb{R}-\{-\rho\} \to \mathbb{R}-\{\gamma\}$ given by

$$f(x) = \frac{\gamma x + 1}{x + \rho}.$$

Show that f is a bijection.

Discussion 2. To show that f is a bijection, we need to show that it is both injective and surjective.

- To show that f is a surjection, we must find for every $a \in \mathbb{R} - \{\gamma\}$ an $x \in \mathbb{R} - \{-\rho\}$ such that $f(x) = a$. In other words, x must satisfy

  $$\frac{\gamma x + 1}{x + \rho} = a.$$

  We can solve for x by taking the following algebraic steps:

  $$\frac{\gamma x + 1}{x + \rho} = a$$

  $$\gamma x + 1 = ax + a\rho$$

  $$\gamma x - ax = a\rho - 1$$

  $$x(\gamma - a) = a\rho - 1$$

  Since $a \neq \gamma$, we can divide by $\gamma - a \neq 0$ to get

  $$x = \frac{a\rho - 1}{\gamma - a}.$$

  We can then algebraically check that our x is a pre-image by showing that $f(x) = a$.

- To show that f is an injection, we will assume that $f(x) = f(y)$ and conclude that $x = y$. Thus,

  $$\frac{\gamma x + 1}{x + \rho} = \frac{\gamma y + 1}{y + \rho}.$$

  We will perform algebra and use the fact that $\gamma\rho \neq 1$ to show that $x = y$.

Proof 2. To prove that f is a bijection, we will prove that it is both an injection and a surjection.

We first prove that f is a surjection. Let $a \in \mathbb{R}-\{\gamma\}$. Thus, $a \neq \gamma$. Consider the real number

$$x = \frac{a\rho - 1}{\gamma - a},$$

which exists since $\gamma \neq a$. Notice that

$$f\left(\frac{a\rho - 1}{\gamma - a}\right) = \frac{\gamma\left(\frac{a\rho-1}{\gamma-a}\right) + 1}{\frac{a\rho-1}{\gamma-a} + \rho} = \frac{\gamma(a\rho - 1) + (\gamma - a)}{a\rho - 1 + \rho(\gamma - a)} =$$

$$\frac{\gamma a\rho - \gamma + \gamma - a}{a\rho - 1 + \gamma\rho - a\rho} = \frac{\gamma a\rho - a}{\gamma\rho - 1} = \frac{a(\gamma\rho - 1)}{\gamma\rho - 1} = a.$$

Thus, every $a \in \mathbb{R} - \gamma$ has a pre-image and f is surjective.

To show that f is injective, we assume that $f(x) = f(y)$. Thus,

$$\frac{\gamma x + 1}{x + \rho} = \frac{\gamma y + 1}{y + \rho}.$$

Performing algebra, we obtain

$$(\gamma x + 1)(y + \rho) = (\gamma y + 1)(x + \rho)$$

$$\gamma xy + \gamma\rho x + y + \rho = \gamma xy + \gamma\rho y + x + \rho$$

$$\gamma xy + \gamma\rho x + y = \gamma xy + \gamma\rho y + x$$

$$\gamma\rho x + y = \gamma\rho y + x$$

$$\gamma\rho x - x = \gamma\rho y - y$$

$$x(\gamma\rho - 1) = y(\gamma\rho - 1)$$

Since $\gamma\rho \neq 1$, then $\gamma\rho - 1 \neq 0$ and we can divide to obtain the desired $x = y$. So, f is injective.

Since f is both injective and surjective, it is a bijection.

□

Question 3. Let S, T, and R be sets, and let $f : S \to T$ and $g : T \to R$ be functions. Show that if $g \circ f$ is injective, then f is injective.

Discussion 3.

`What we know:` $g \circ f$ is injective. Thus, anytime that $g(f(x)) = g(f(y))$, we can conclude that $x = y$.

`What we want:` f is injective. Thus, we will assume that $f(x) = f(y)$; we need to conclude that $x = y$.

`What we'll do:` We will start off assuming that $f(x) = f(y)$. We will then apply the function g to both sides and then use the injectivity of $g \circ f$.

Proof 3. Assume that $f(x) = f(y)$. Applying the function g to both sides, we get that $g(f(x)) = g(f(y))$. Since $g \circ f$ is injective and $g(f(x)) = g(f(y))$, we can conclude that $x = y$. Thus, f is injective.

Question 4. Let $C([0, 1])$ be the set of all real, continuous functions on the interval $[0, 1]$. That is,

$$C([0,1]) = \{f \mid f : [0,1] \to \mathbb{R} \text{ is a continuous function}\}.$$

Thus, an element of the set $C([0, 1])$ is simply a function $f(x)$ that is continuous on $[0, 1]$. Furthermore, consider the function $\varphi : C([0, 1]) \to \mathbb{R}$ given by

$$\varphi(f) = \int_0^1 f(x)\, dx.$$

(a) Show that the function φ is surjective by showing that for every $a \in \mathbb{R}$, there exists a pre-image $f \in C([0,1])$ such that $\varphi(f) = a$.

(b) Show that the function φ is not injective by finding two distinct functions $f, g \in C([0,1])$ such that $\varphi(f) = \varphi(g)$.

Discussion 4a. We wish to show that for any $a \in \mathbb{R}$, there exists a function $f(x)$ such that

$$\varphi(f) = \int_0^1 f(x)\,dx = a.$$

Many examples will work, but the simplest is the constant function $f(x) = a$.

Proof 4a. Let $a \in \mathbb{R}$. We will show that φ is a surjection by showing that a has a pre-image. Consider $f(x) = a$, the constant function. Notice that

$$\varphi(f(x)) = \varphi(a) = \int_0^1 a\,dx = a(1-0) = a.$$

Thus, f is surjective.

$\square$

Discussion 4b. To show that f is not injective, we must find two distinct continuous function f and g such that $\varphi(f) = \varphi(g)$ and thus

$$\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx.$$

We can choose a simple function, like $f(x) = x$ and compute that

$$\varphi(f) = \int_0^1 f(x)\,dx = \int_0^1 x\,dx = \frac{1}{2}.$$

Notice that the constant function $g(x) = \frac{1}{2}$ also has the same integral. Thus, these two function $f(x) \neq g(x)$ are mapped to the same real value.

Proof 4b. Consider the continuous functions $f(x) = x$ and $g(x) = \frac{1}{2}$, the constant function. Notice that

$$\varphi(f(x)) = \int_0^1 x\,dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$$

and that

$$\varphi(g(x)) = \varphi\left(\frac{1}{2}\right) = \int_0^1 \frac{1}{2}\,dx = \frac{1}{2}(1-0) = \frac{1}{2}.$$

Thus, $f \neq g$, but $\varphi(f) = \varphi(g)$ and thus φ is not injective.

$\square$